Global existence and stability of solutions for Kirchhoff-type parabolic system with logarithmic source term

Tuğrul Cömert[1] and Erhan Pişkin[2]

[1]Dicle University, Department of Mathematics, Diyarbakır, Turkey
[2]Dicle University, Department of Mathematics, Diyarbakır, Turkey
E-mail: tugrulcomertt@gmail.com[1], episkin@dicle.edu.tr[2]

Abstract

This paper deals with the initial-boundary value problem for Kirchhoff-type parabolic system with logarithmic source term. We discuss the global existence and exponential energy decay estimates of weak solutions under some conditions by employing potential method.

2010 Mathematics Subject Classification. **35A01**. 35B40, 35K55.
Keywords. Kirchhoff type parabolic system, global existence, logarithmic nonlinearity.

1 Introduction

In this paper, we investigate the global existence and decay of solutions for the Kirchhoff type parabolic system with logarithmic source term

$$\begin{cases} u_t - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta u - \Delta u_t = |u|^{q-2} u \ln |u|, & x \in \Omega, \quad t > 0, \\ v_t - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta v - \Delta v_t = |v|^{q-2} v \ln |v|, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, \ v(x,t) = 0 & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bound domain in $\mathbb{R}^n$ with smooth boundary, $M(s) = 1 + s^\gamma$, $(\gamma > 0)$ and $2\gamma + 2 \leq q$.

Studies of logarithmic nonlinearity have a long history in physics as it occurs naturally in different areas of physics such as supersymmetric field theories, inflation cosmology, nuclear physics and quantum mechanics [1, 2].

When $M \equiv 1$ and $q = 2$, the system (1.1) become semilinear pseudo-parabolic equation as follow

$$u_t - \Delta u - \Delta u_t = u \ln |u|. \tag{1.2}$$

Chen and Tian [3] obtained the global existence, behavior of vacuum isolation and blow-up of solutions at $+\infty$ of the equation (1.2). Without Δu_t, the equation (1.2) become the following semilinear parabolic equation

$$u_t - \Delta u = u \ln |u|. \tag{1.3}$$

Chen et al. [4] studied the global existence, decay and blow-up at $+\infty$ of solutions of the equation (1.3).

When $M \equiv 1$ and $q > 2$, Peng and Zhou [5] investigated the following parabolic equation with logarithmic source term

$$u_t - \Delta u = |u|^{q-2} u \ln |u|.$$

Advanced Studies: Euro-Tbilisi Mathematical Journal Special Issue (10 - 2022), pp. 153–170.

Tbilisi Centre for Mathematical Sciences.
Received by the editors: 17 May 2021.
Accepted for publication: 29 October 2021.

They studied the global existence and blow-up of solutions. Also, they discussed the upper bound of blow-up time under suitable conditions.

Nhan and Truong [6] studied the following nonlinear pseudo-parabolic equation

$$u_t - \Delta u_t - div\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{q-2} u \ln |u|.$$

They studied results as regard the existence or non-existence of global solutions. Also, He et al. [7] investigated the decay and the finite time blow-up for solutions of the equation.

Cao and Liu [8] studied the following nonlinear evolution equation with logarithmic source term

$$u_t - div\left(|\nabla u|^{p-2}\nabla u\right) - k\Delta u_t = |u|^{q-2} u \ln |u|.$$

They established the existence of global weak solutions. Moreover, they considered global boundedness and blowing-up at ∞.

Pişkin and Cömert [9] studied the following Kirchhoff type parabolic equation

$$u_t - M(\|\nabla u\|^2)\Delta u - \Delta u_t = |u|^{q-2} u \ln |u|.$$

They studied the finite time blow-up for weak solutions by employing potential well method and concavity method.

Yang et al. [10] considered the following equation

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + |u_t|^{p-1} u_t - \Delta u_t = u^{q-1}\ln |u|, \tag{1.4}$$

where $M(s) = \alpha + \beta s^\gamma$, $\gamma > 0, \alpha \geq 1, \beta > 0$. They studied existence finite time blow up of solutions.

Wang et. al [11] investigated the following Kirchhoff type system with logarithmic source term

$$\begin{cases} u_{tt} + M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta u + u_t = |u|^{q-2} u \ln |u|, \\ v_{tt} + M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta v + v_t = |v|^{q-2} v \ln |v|, \end{cases}$$

with $M(s) = \alpha + \beta s^\gamma$, $\gamma > 0, \alpha \geq 1, \beta \geq 0$ and $q \geq 2\gamma + 2$. They studied global existence and finite time blow up under the different conditions by employing potential well method and concavity method.

Recently many other authors investigated parabolic or hyperbolic type equations with logarithmic nonlinearity (see [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]).

Motivated by the above studies, in this work we investigate the global existence and decay estimate of solutions for the Eq. (1.1).

Our plan in this paper is as follows: In Section 2, we discussed some lemmas which will be needed later. In Section 3, we proved that the global existence and exponential decay of solutions.

2 Preliminaries

For simplicity, we denote

$$\|u\| = \|u\|_{L^2(\Omega)}, \ \ \|u\|_s = \|u\|_{L^s(\Omega)}, \ \ \|u\|_{H_0^1(\Omega)} = \left(\|u\|^2 + \|\nabla u\|^2\right)^{\frac{1}{2}},$$

for $1 < s < \infty$.

For $H_0^1(\Omega) \times H_0^1(\Omega)\backslash\{0\}$, we define the energy functional

$$\begin{aligned} J(u,v) &= \frac{1}{2}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \frac{1}{2(\gamma+1)}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\ &\quad -\frac{1}{q}\left(\int_\Omega |u|^q \ln|u|\,dx + \int_\Omega |v|^q \ln|v|\,dx\right) + \frac{1}{q^2}\left(\|u\|_q^q + \|v\|_q^q\right), \end{aligned} \tag{2.1}$$

and Nehari functional

$$\begin{aligned} I(u,v) &= \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\ &\quad -\left(\int_\Omega |u|^q \ln|u|\,dx + \int_\Omega |v|^q \ln|v|\,dx\right). \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), we get

$$\begin{aligned} J(u,v) &= \frac{1}{q}I(u,v) + \frac{q-2}{2q}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &\quad +\frac{q-2\gamma-2}{2q\gamma+2q}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} + \frac{1}{q^2}\left(\|u\|_q^q + \|v\|_q^q\right). \end{aligned} \tag{2.3}$$

Let

$$\mathcal{N} = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)\backslash\{0\} : I(u,v) = 0\},$$

be the Nehari manifold. Also, we may define

$$d = \inf_{(u,v)\in\mathcal{N}} J(u,v), \tag{2.4}$$

and

$$W = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)\backslash\{0\} : J(u,v) < d,\ I(u,v) > 0\}.$$

We refer to W as the potential well and d as the depth of the well.

Lemma 2.1. $J(u,v)$ is a nonincreasing function for $t \geq 0$ and

$$\frac{d}{dt}J(u,v) = -\left(\|u_t\|^2 + \|v_t\|^2\right) - \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \leq 0. \tag{2.5}$$

Proof. Multiplying the first equation (1.1) by u_t and the second equation (1.1) by v_t, and integrating on Ω, we have

$$\begin{aligned} &\int_\Omega |u_t|^2\,dx + \frac{1}{2}\left[1 + \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma}\right]\frac{d}{dt}\|\nabla u\|^2 + \int_\Omega |\nabla u_t|^2\,dx \\ = &\ \frac{d}{dt}\left(\frac{1}{q}\int_\Omega |u|^q \ln|u|\,dx - \frac{1}{q^2}\int_\Omega |u|^q\,dx\right), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} & \int_\Omega |v_t|^2\,dx + \frac{1}{2}\left[1+\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^\gamma\right]\frac{d}{dt}\|\nabla v\|^2 + \int_\Omega |\nabla v_t|^2\,dx \\ = & \ \frac{d}{dt}\left(\frac{1}{q}\int_\Omega |v|^q \ln|v|\,dx - \frac{1}{q^2}\int_\Omega |v|^q\,dx\right). \end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$\begin{aligned} & \int_\Omega \left(|u_t|^2+|v_t|^2\right)dx + \int_\Omega\left(|\nabla u_t|^2+|\nabla v_t|^2\right)dx \\ & +\frac{1}{2}\left[1+\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^\gamma\right]\frac{d}{dt}\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ = & \ \frac{d}{dt}\left(\frac{1}{q}\int_\Omega \left(|u|^q\ln|u|+|v|^q\ln|v|\right)dx - \frac{1}{q^2}\int_\Omega\left(|u|^q+|v|^q\right)dx\right). \end{aligned} \tag{2.8}$$

Thus, by (2.8), we obtain

$$\begin{aligned} & \frac{d}{dt}\left(\frac{1}{2}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\frac{1}{2(\gamma+1)}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1}\right) \\ & -\frac{d}{dt}\left(\frac{1}{q}\left(\int_\Omega |u|^q\ln|u|\,dx+\int_\Omega |v|^q\ln|v|\,dx\right)-\frac{1}{q^2}\left(\|u\|_q^q+\|v\|_q^q\right)\right) \\ = & \ -\left(\|u_t\|^2+\|v_t\|^2\right)-\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right), \end{aligned}$$

that is

$$\frac{d}{dt}J(u,v) = -\left(\|u_t\|^2+\|v_t\|^2\right)-\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\leq 0.$$

Moreover, integrating (2.8) with respect to t on $[0,t]$, we arrive at

$$\begin{aligned} & \int_0^t\left(\|u_\tau\|^2+\|v_\tau\|^2\right)d\tau+\int_0^t\left(\|\nabla u_\tau\|^2+\|\nabla v_\tau\|^2\right)d\tau \\ & +\frac{1}{2}\int_0^t\left[1+\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^\gamma\right]d\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ = & \ \frac{1}{q}\int_\Omega\left(|u|^q\ln|u|+|v|^q\ln|v|\right)dx-\frac{1}{q^2}\left(\|u\|_q^q+\|v\|_q^q\right) \\ & -\frac{1}{q}\int_\Omega\left(|u_0|^q\ln|u_0|+|v_0|^q\ln|v_0|\right)dx+\frac{1}{q^2}\left(\|u_0\|_q^q+\|v_0\|_q^q\right). \end{aligned} \tag{2.9}$$

We deal with the third term in the left hand of (2.9) as follows

$$\begin{aligned} & \frac{1}{2}\int_0^t\left[1+\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^\gamma\right]d\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ = & \ \frac{1}{2}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-\frac{1}{2}\left(\|\nabla u_0\|^2+\|\nabla v_0\|^2\right) \\ & +\frac{1}{2(\gamma+1)}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1}-\frac{1}{2(\gamma+1)}\left(\|\nabla u_0\|^2+\|\nabla v_0\|^2\right)^{\gamma+1}. \end{aligned} \tag{2.10}$$

Inserting (2.10) into (2.9), and obtain

$$\begin{aligned}
&\int_0^t \left(\|u_\tau\|^2 + \|v_\tau\|^2\right) d\tau + \int_0^t \left(\|\nabla u_\tau\|^2 + \|\nabla v_\tau\|^2\right) d\tau + \frac{1}{2}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\
&+\frac{1}{2(\gamma+1)}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} - \frac{1}{q}\int_\Omega \left(|u|^q \ln|u| + |v|^q \ln|v|\right) dx + \frac{1}{q^2}\left(\|u\|_q^q + \|v\|_q^q\right) \\
=\ & \frac{1}{2}\left(\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right) + \frac{1}{2(\gamma+1)}\left(\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right)^{\gamma+1} \\
&-\frac{1}{q}\int_\Omega \left(|u_0|^q \ln|u_0| + |v_0|^q \ln|v_0|\right) dx + \frac{1}{q^2}\left(\|u_0\|_q^q + \|v_0\|_q^q\right), \qquad (2.11)
\end{aligned}$$

that is

$$\int_0^t \left(\|u_\tau\|^2 + \|v_\tau\|^2\right) d\tau + \int_0^t \left(\|\nabla u_\tau\|^2 + \|\nabla v_\tau\|^2\right) d\tau + J(u,v) = J\left(u_0, v_0\right). \qquad (2.12)$$

Q.E.D.

Lemma 2.2. Let $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)\backslash\{0\}$, we consider the function $j : \lambda \to J(\lambda u, \lambda v)$ for $\lambda > 0$. Then we possess

(i) $\lim_{\lambda\to 0^+} j(\lambda) = 0$ and $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$;

(ii) there is a unique $\lambda^* > 0$ such that $j'(\lambda^*) = 0$;

(iii) $j(\lambda)$ is increasing on $(0,\lambda^*)$, decreasing on $(\lambda^*,+\infty)$ and taking the maximum at λ^*; $I(\lambda u) = \lambda j'(\lambda)$ and

$$I(\lambda u)\begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < +\infty. \end{cases}$$

Proof. By the definition of j, for $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)\backslash\{0\}$, we get

$$\begin{aligned}
j(\lambda) =\ & \frac{1}{2}\left(\|\nabla(\lambda u)\|^2 + \|\nabla(\lambda v)\|^2\right) + \frac{1}{2(\gamma+1)}\left(\|\nabla(\lambda u)\|^2 + \|\nabla(\lambda v)\|^2\right)^{\gamma+1} \\
&-\frac{1}{q}\left(\int_\Omega |\lambda u|^q \ln|\lambda u|\, dx + \int_\Omega |\lambda v|^q \ln|\lambda v|\, dx\right) + \frac{1}{q^2}\left(\|\lambda u\|_q^q + \|\lambda v\|_q^q\right) \\
=\ & \frac{\lambda^2}{2}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \frac{\lambda^{2(\gamma+1)}}{2(\gamma+1)}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&-\frac{\lambda^q}{q}\left(\int_\Omega |u|^q \ln|u|\, dx + \int_\Omega |v|^q \ln|v|\, dx\right) - \frac{\lambda^q}{q}\ln\lambda\left(\|u\|_q^q + \|v\|_q^q\right) \\
&+\frac{\lambda^q}{q^2}\left(\|u\|_q^q + \|v\|_q^q\right). \qquad (2.13)
\end{aligned}$$

We see that *(i)* holds due to $\int_\Omega (|u|^q + |v|^q)\,dx \neq 0$. We have

$$\begin{aligned}
\frac{d}{d\lambda}j(\lambda) &= \lambda\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \lambda^{2\gamma+1}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&\quad -\lambda^{q-1}\left(\int_\Omega |u|^q \ln|u|\,dx + \int_\Omega |v|^q \ln|v|\,dx\right) - \lambda^{q-1}\ln\lambda\left(\|u\|_q^q + \|v\|_q^q\right) \\
&\quad -\frac{\lambda^{q-1}}{q}\left(\|u\|_q^q + \|v\|_q^q\right) + \frac{\lambda^{q-1}}{q}\left(\|u\|_q^q + \|v\|_q^q\right) \\
&= \lambda\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \lambda^{2\gamma+1}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&\quad -\lambda^{q-1}\left(\int_\Omega |u|^q \ln|u|\,dx + \int_\Omega |v|^q \ln|v|\,dx\right) - \lambda^{q-1}\ln\lambda\left(\|u\|_q^q + \|v\|_q^q\right).
\end{aligned}$$

Let $\varphi(\lambda) = \lambda^{-1}j'(\lambda)$, thus we obtain

$$\begin{aligned}
\varphi(\lambda) &= \lambda^{-1}j'(\lambda) \\
&= \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \lambda^{2\gamma}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&\quad -\lambda^{q-2}\left(\int_\Omega |u|^q \ln|u|\,dx + \int_\Omega |v|^q \ln|v|\,dx\right) - \lambda^{q-2}\ln\lambda\left(\|u\|_q^q + \|v\|_q^q\right).
\end{aligned}$$

Then

$$\begin{aligned}
\varphi'(\lambda) &= 2\gamma\lambda^{2\gamma-1}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} - (q-2)\lambda^{q-3}\left(\int_\Omega |u|^q \ln|u|\,dx + \int_\Omega |v|^q \ln|v|\,dx\right) \\
&\quad -(q-2)\lambda^{q-3}\ln\lambda\left(\|u\|_q^q + \|v\|_q^q\right) - \lambda^{q-3}\left(\|u\|_q^q + \|v\|_q^q\right),
\end{aligned}$$

which yields that there exists a $\lambda^* > 0$ such that $\varphi'(\lambda) > 0$ on $(0,\lambda^*)$, $\varphi'(\lambda) < 0$ on $(\lambda^*, +\infty)$ and on $\varphi'(\lambda) = 0$. So, $\varphi(\lambda)$ is increasing on $(0,\lambda^*)$, decreasing on $(\lambda^*, +\infty)$. Since $\lim_{\lambda\to 0^+}\varphi(\lambda) = \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) > 0$, $\lim_{\lambda\to+\infty}\varphi(\lambda) = -\infty$, there exists a unique $\lambda^* > 0$ such that $\varphi(\lambda^*) = 0$, i.e., $j'(\lambda^*) = 0$. So *(ii)* holds. Then, $j'(\lambda) = \lambda\varphi(\lambda)$ is positive on $(0,\lambda^*)$, negative on $(\lambda^*, +\infty)$. Thus, $j(\lambda)$ is increasing on $(0,\lambda^*)$, decreasing on $(\lambda^*, +\infty)$ and taking the maximum at λ^*. From (2.2), we get

$$\begin{aligned}
I(\lambda u, \lambda v) &= \left(\|\nabla(\lambda u)\|^2 + \|\nabla(\lambda v)\|^2\right) + \left(\|\nabla(\lambda u)\|^2 + \|\nabla(\lambda v)\|^2\right)^{\gamma+1} \\
&\quad -\left(\int_\Omega |\lambda u|^q \ln|\lambda u|\,dx + \int_\Omega |\lambda v|^q \ln|\lambda v|\,dx\right) \\
&= \lambda^2\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \lambda^{2(\gamma+1)}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&\quad -\lambda^q\left(\int_\Omega |u|^q \ln|u|\,dx + \int_\Omega |v|^q \ln|v|\,dx\right) - \lambda^q\ln\lambda\left(\|u\|_q^q + \|v\|_q^q\right) \\
&= \lambda j'(\lambda).
\end{aligned}$$

Thus, $I(\lambda u, \lambda v) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u, \lambda v) < 0$ for $\lambda^* < \lambda < +\infty$ and $I(\lambda^* u, \lambda^* v) = 0$. So *(iii)* holds. For this reason, the proof is completed. Q.E.D.

Lemma 2.3. d defined by (2.4) is positive and there exists a positive function $u, v \in \mathcal{N}$ such that $J(u, v) = d$.

Proof. Let $\{u_k, v_k\}_k^\infty \subset \mathcal{N}$ be a minimizing sequence for J, which means that

$$\lim_{k\to\infty} J(u_k, v_k) = d. \tag{2.14}$$

We can easy show that $\{|u_k, v_k|\}_k \subset \mathcal{N}$ is also a minimizing sequence for J due to $|u_k, v_k| \in \mathcal{N}$ and $J(|u_k, v_k|) = J(u_k, v_k)$. Therefore, we can suppose that $u_k, v_k > 0$ a.e. Ω for all $k \in N$.

Moreover, we have already observed that J is coercive on $\mathcal{N}$ which satisfies that $\{u_k, v_k\}_k^\infty$ is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$. Let $\mu > 0$ be small enough such that $q + \mu < \frac{2n}{n-2}$. Since $H_0^1(\Omega) \hookrightarrow L^{q+\mu}(\Omega)$ is compact, so there exists a function u and a subsequence of $\{u_k, v_k\}_k^\infty$, still denote by $\{u_k, v_k\}_k^\infty$, such that

$$\begin{aligned}
u_k \to u &\quad \text{weakly in} \quad H_0^1(\Omega),\\
v_k \to v &\quad \text{weakly in} \quad H_0^1(\Omega),\\
u_k \to u &\quad \text{strongly in} \quad L^{q+\mu}(\Omega),\\
v_k \to v &\quad \text{strongly in} \quad L^{q+\mu}(\Omega),\\
u_k(x) \to u(x) &\quad \text{a.e. in} \quad \Omega,\\
v_k(x) \to v(x) &\quad \text{a.e. in} \quad \Omega.
\end{aligned}$$

Also, $u, v \geq 0$ a.e. in Ω. First, we prove $u, v \neq 0$. From the dominated convergence theorem, we get

$$\int_\Omega |u|^q \ln|u|\, dx = \lim_{k\to\infty} \int_\Omega |u_k|^q \ln|u_k|\, dx, \tag{2.15}$$

$$\int_\Omega |v|^q \ln|v|\, dx = \lim_{k\to\infty} \int_\Omega |v_k|^q \ln|v_k|\, dx,$$

and

$$\int_\Omega |u|^q\, dx = \lim_{k\to\infty} \int_\Omega |u_k|^q\, dx, \tag{2.16}$$

$$\int_\Omega |v|^q\, dx = \lim_{k\to\infty} \int_\Omega |v_k|^q\, dx.$$

From the weak lower semicontinuity of $H_0^1(\Omega) \times H_0^1(\Omega)$, we get

$$\|\nabla u\|^2 + \|\nabla v\|^2 \leq \liminf_{k\to\infty} \left(\|\nabla u\|^2 + \|\nabla v\|^2\right), \tag{2.17}$$

$$\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \leq \lim_{k\to\infty} \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1}.$$

Using (2.1), (2.14), (2.15), (2.16) and (2.17), we have

$$\begin{aligned} J(u,v) &= \frac{1}{2}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\frac{1}{2(\gamma+1)}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} \\ &\quad -\frac{1}{q}\left(\int_\Omega |u|^q\ln|u|\,dx+\int_\Omega |v|^q\ln|v|\,dx\right)+\frac{1}{q^2}\left(\|u\|_q^q+\|v\|_q^q\right) \\ &\leq \liminf_{k\to\infty}\frac{1}{2}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\lim_{k\to\infty}\frac{1}{2(\gamma+1)}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} \\ &\quad -\lim_{k\to\infty}\frac{1}{q}\left(\int_\Omega |u|^q\ln|u|\,dx+\int_\Omega |v|^q\ln|v|\,dx\right)+\lim_{k\to\infty}\frac{1}{q^2}\left(\|u\|_q^q+\|v\|_q^q\right) \\ &= \liminf_{k\to\infty} J(u_k)=d. \end{aligned} \tag{2.18}$$

Using (2.2), (2.15) and (2.17), we have

$$\begin{aligned} I(u,v) &= \left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} \\ &\quad -\left(\int_\Omega |u|^q\ln|u|\,dx+\int_\Omega |v|^q\ln|v|\,dx\right) \\ &\leq \liminf_{k\to\infty}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\lim_{k\to\infty}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} \\ &\quad -\lim_{k\to\infty}\left(\int_\Omega |u|^q\ln|u|\,dx+\int_\Omega |v|^q\ln|v|\,dx\right) \\ &= \liminf_{k\to\infty} I(u_k)=0. \end{aligned} \tag{2.19}$$

Since $(u_k,v_k)\in\mathcal{N}$, we have $I(u_k,v_k)=0$. So, by employing the fact $\alpha^{-\mu}\ln\alpha\leq(e\mu)^{-1}$ for $\alpha\geq 1$ and the Sobolev embedding inequality, we get

$$\begin{aligned} \left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} &= \int_\Omega |u|^q\ln|u|\,dx+\int_\Omega |v|^q\ln|v|\,dx \\ &\leq (e\mu)^{-1}\left(\int_\Omega |u_k|^{q+\mu}\,dx+\int_\Omega |v_k|^{q+\mu}\,dx\right) \\ &\leq (e\mu)^{-1}\left(\|u_k\|_{q+\mu}^{q+\mu}+\|v_k\|_{q+\mu}^{q+\mu}\right) \\ &\leq C\left(\|\nabla u_k\|_2^{q+\mu}+\|\nabla v_k\|_2^{q+\mu}\right), \end{aligned}$$

where C is positive and Sobolev embedding constant. This yields that

$$\int_\Omega |u_k|^q\ln|u_k|\,dx+\int_\Omega |v_k|^q\ln|v_k|\,dx=\|\nabla u_k\|^2+\|\nabla v_k\|^2\geq C. \tag{2.20}$$

By (2.20) and (2.15), we obtain

$$\int_\Omega |u|^q\ln|u|\,dx+\int_\Omega |v|^q\ln|v|\,dx\geq C.$$

Thus, we have $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \backslash \{0\}$.

If $I(u_k, v_k) < 0$, by Lemma 2.2, there exists a λ^* such that $I(\lambda^* u, \lambda^* v) = 0$ and $0 < \lambda^* < 1$. Thus, $(\lambda^* u, \lambda^* v) \in \mathcal{N}$. By (2.3), (2.4), (2.16) and (2.17), we get

$$\begin{aligned}
d &\leq J(\lambda^* u, \lambda^* v) \\
&= \frac{1}{q} I(\lambda^* u, \lambda^* v) + \frac{q-2}{2q} \left(\|\nabla (\lambda^* u)\|^2 + \|\nabla (\lambda^* v)\|^2 \right) \\
&\quad + \frac{q-2\gamma-2}{2q\gamma+2q} \left(\|\nabla (\lambda^* u)\|^2 + \|\nabla (\lambda^* v)\|^2 \right)^{\gamma+1} + \frac{1}{q^2} \left(\|\lambda^* u\|_q^q + \|\lambda^* v\|_q^q \right) \\
&= \frac{q-2}{2q} \left(\|\nabla (\lambda^* u)\|^2 + \|\nabla (\lambda^* v)\|^2 \right) + \frac{q-2\gamma-2}{2q\gamma+2q} \left(\|\nabla (\lambda^* u)\|^2 + \|\nabla (\lambda^* v)\|^2 \right)^{\gamma+1} \\
&\quad + \frac{1}{q^2} \left(\|\lambda^* u\|_q^q + \|\lambda^* v\|_q^q \right) \\
&\leq (\lambda^*)^2 \left[\frac{q-2}{2q} \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right) + \frac{q-2\gamma-2}{2q\gamma+2q} \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)^{\gamma+1} \right] \\
&\quad + (\lambda^*)^2 \frac{1}{q^2} \left(\|u\|_q^q + \|v\|_q^q \right) \\
&\leq (\lambda^*)^2 \liminf_{k\to\infty} \left[\frac{q-2}{2q} \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right) + \frac{q-2\gamma-2}{2q\gamma+2q} \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)^{\gamma+1} \right] \\
&\quad + (\lambda^*)^2 \liminf_{k\to\infty} \frac{1}{q^2} \left(\|u\|_q^q + \|v\|_q^q \right) \\
&= (\lambda^*)^2 \liminf_{k\to\infty} J(u_k) \\
&= (\lambda^*)^2 d,
\end{aligned}$$

which indicates $\lambda^* \geq 1$ by $d > 0$. It contradicts $0 < \lambda^* < 1$. Then, by (2.19), we have $I(u,v) = 0$. For this reason, $u, v \in \mathcal{N}$. By (2.14), we have $J(u,v) \geq d$. By (2.18), we have $J(u,v) \leq d$. So, $J(u,v) = d$. Q.E.D.

Lemma 2.4. [6] For any $u \in H_0^1(\Omega)$, $p \geq 1$, and $r \geq 1$, the inequality

$$\|u\|_q \leq C \|\nabla u\|_p^\theta \|u\|_r^{1-\theta},$$

is valid, where

$$\theta = \left(\frac{1}{r} - \frac{1}{q} \right) \left(\frac{1}{n} - \frac{1}{p} + \frac{1}{r} \right)^{-1},$$

and

-for $p \geq n = 1$, $r \leq q \leq \infty$;

-for $n > 1$ and $p < n$, $q \in [r, p_*]$ if $r \leq p_*$ and $q \in [p_*, r]$ if $r \geq p_*$;

-for $p = n > 1$, $r \leq q < \infty$;

-for $p > n > 1$, $r \leq q \leq \infty$.

Here, the constant C depends on n, p, q and r.

Lemma 2.5. [25] Let $f : R^+ \to R^+$ be a nonincreasing function and σ is a nonnegative constant such that
$$\int_t^{+\infty} f^{1+\sigma}(s)ds \leq \frac{1}{\omega} f^\sigma(0) f(t), \quad \forall t \geq 0.$$

Hence

(a) $f(t) \leq f(0)e^{1-\omega t}$, for all $t \geq 0$, whenever $\sigma = 0$,

(b) $f(t) \leq f(0) \left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}}$, for all $t \geq 0$, whenever $\sigma > 0$.

3 Main results

Definition 3.1. (Weak solution). We say that function $(u(t), v(t))$ is weak solutions of the problem (1.1) over $\Omega\times[0,T]$, if $(u,v) \in L^\infty(0,T; H_0^1(\Omega))\times L^\infty(0,T; H_0^1(\Omega))$ with $(u_t, v_t) \in L^2(0,T; H_0^1(\Omega))\times L^2(0,T; H_0^1(\Omega))$ and satisfies the initial condition $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)\backslash\{0\}$, and the following equality

$$\langle u_t, \varphi\rangle + \langle \nabla u, \nabla\varphi\rangle + \left\langle \|\nabla u\|^{2\gamma} \nabla u, \nabla\varphi \right\rangle + \langle \nabla u_t, \nabla\varphi\rangle = \int_\Omega |u|^{q-2} u \ln|u| \varphi dx,$$

and

$$\langle v_t, \psi\rangle + \langle \nabla v, \nabla\psi\rangle + \left\langle \|\nabla v\|^{2\gamma} \nabla v, \nabla\psi \right\rangle + \langle \nabla v_t, \nabla\psi\rangle = \int_\Omega |v|^{q-2} v \ln|v| \psi dx,$$

for all $(\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ holds for a.e. $t \in [0,T]$, and $\langle ., . \rangle$ means the inner product in space $L^2(\Omega)$; that is
$$\langle \eta, \xi\rangle = \int_\Omega \eta(x)\xi(x)dx.$$

Definition 3.2. (Maximal Existence Time). Suppose that u be weak solutions of problem (1.1). We define the maximal existence time $T_{\max}$ as follows

$$T_{\max} = \sup\{T > 0 : u(t) \text{ exists on } [0,T]\}.$$

Then

(i) If $T_{\max} < \infty$, we say that u blows up in finite time and $T_{\max}$ is the blow-up time;

(ii) If $T_{\max} = \infty$, we say that u is global.

Theorem 3.3. (Global Existence). Let $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Then there is a unique global weak solution (u, v) of (1.1) satisfying (u_0, v_0). We have $(u(t), v(t)) \in \mathcal{W}$ holds for all $0 \leq t < +\infty$, and the energy estimate

$$\int_0^t \left(\|u_\tau(\tau)\|^2_{H_0^1(\Omega)} + \|v_\tau(\tau)\|^2_{H_0^1(\Omega)} \right) d\tau + J(u,v) \leq J(u_0, v_0), \quad 0 \leq t \leq +\infty.$$

Also, the solution decay exponentially provided $(u_0, v_0) \in \mathcal{W}$.

Proof. Let $(u_0, v_0) \in \mathcal{W}$ be the initial data. Let $\{w_j\}_{j=1}^{\infty}$ be a system of Galerkin base function in space $H_0^1(\Omega) \times H_0^1(\Omega)$. Construct the approximate solution of $(u_m(x,t), v_m(x,t))$ of the problem (1.1). Let

$$u_m(x,t) = \sum_{j=1}^{m} a_{mj}(t) w_j(x), \tag{3.1}$$

and

$$v_m(x,t) = \sum_{j=1}^{m} b_{mj}(t) w_j(x),$$

satisfying

$$\begin{aligned} & \int_\Omega u_{mt} w_i dx + \int_\Omega \nabla u_m \nabla w_i dx \\ & + \int_\Omega \left(\|\nabla u_m\|^2 + \|\nabla v_m\|^2 \right)^\gamma \nabla u_m \nabla w_i dx \ + \int_\Omega \nabla u_{mt} \nabla w_i dx \\ = \ & \int_\Omega |u_m|^{q-2} u_m \ln |u_m| \, w_i dx, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} & \int_\Omega v_{mt} w_i dx + \int_\Omega \nabla v_m \nabla w_i dx \\ & + \int_\Omega \left(\|\nabla u_m\|^2 + \|\nabla v_m\|^2 \right)^\gamma \nabla v_m \nabla w_i dx \ + \int_\Omega \nabla v_{mt} \nabla w_i dx \\ = \ & \int_\Omega |v_m|^{q-2} v_m \ln |v_m| \, w_i dx, \end{aligned} \tag{3.3}$$

for $i \in \{1, 2, ..., m\}$. And as $m \to \infty$, we get

$$u_{0m} = \sum_{j=1}^{m} a_{mj} w_j \to u_0 \quad \text{strongly in } H_0^1(\Omega), \tag{3.4}$$

and

$$v_{0m} = \sum_{j=1}^{m} b_{mj} w_j \to v_0 \ \text{ strongly in } H_0^1(\Omega). \tag{3.5}$$

Multiplying (3.2) by $h'_{mj}(t)$, and (3.3) by $g'_{mj}(t)$, summing for $i = 1, 2, ..., m$, we have

$$\begin{aligned} & \int |u_{mt}|^2 dx + \frac{1}{2} \left[1 + \left(\|\nabla u_m\|^2 + \|\nabla v_m\|^2 \right)^\gamma \right] \frac{d}{dt} \|\nabla u_m\|^2 + \int |\nabla u_{mt}|^2 dx \\ = \ & \frac{d}{dt} \left(\frac{1}{q} \int_\Omega |u_m|^q \ln |u_m| \, dx - \frac{1}{q^2} \int_\Omega |u_m|^q \, dx \right), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} & \int |v_{mt}|^2 dx + \frac{1}{2} \left[1 + \left(\|\nabla u_m\|^2 + \|\nabla v_m\|^2 \right)^\gamma \right] \frac{d}{dt} \|\nabla v_m\|^2 + \int |\nabla v_{mt}|^2 dx \\ = \ & \frac{d}{dt} \left(\frac{1}{q} \int_\Omega |v_m|^q \ln |v_m| \, dx - \frac{1}{q^2} \int_\Omega |v_m|^q \, dx \right). \end{aligned} \tag{3.7}$$

Integrating (3.6) and (3.7) over $(0,t)$ and then summarizing the obtained results, for $0 \leq t < \infty$, we obtain

$$\int_0^t \left(\|u_{m\tau}\|^2 + \|v_{m\tau}\|^2 \right) d\tau + \int_0^t \left(\|\nabla u_{m\tau}\|^2 + \|\nabla v_{m\tau}\|^2 \right) d\tau + J(u_m, v_m) = J(u_{0m}, v_{0m}).$$

Recalling (3.4) and (3.5) yields $J(u_{0m}, v_{0m}) \to J(u_0, v_0)$ as $m \to \infty$, which says that for sufficiently large m

$$\int_0^t \left(\|u_{m\tau}(\tau)\|^2_{H_0^1(\Omega)} + \|v_{m\tau}(\tau)\|^2_{H_0^1(\Omega)} \right) d\tau + J(u_m, v_m) < d. \tag{3.8}$$

Then from (2.3) it gives

$$\begin{aligned} J(u_m, v_m) &= \frac{1}{q} I(u_m, v_m) + \frac{q-2}{2q} \left(\|\nabla u_m\|^2 + \|\nabla v_m\|^2 \right) \\ &\quad + \frac{q-2\gamma-2}{2q\gamma+2q} \left(\|\nabla u_m\|^2 + \|\nabla v_m\|^2 \right)^{\gamma+1} + \frac{1}{q^2} \left(\|u_m\|_q^q + \|v_m\|_q^q \right). \end{aligned}$$

(3.4) and (3.5), for sufficiently large m and $0 \leq t < \infty$, we get $(u_{0m}, v_{0m}) \in \mathcal{W}$. From (3.8), we have that $(u_m(t), v_m(t)) \in \mathcal{W}$ for sufficiently large m and $0 \leq t < \infty$. Thus for $0 \leq t < \infty$ and $q \geq 2\gamma + 2$, (3.8) gives

$$\begin{aligned} &\int_0^t \left(\|u_{m\tau}\|^2 + \|v_{m\tau}\|^2 \right) d\tau + \int_0^t \left(\|\nabla u_{m\tau}\|^2 + \|\nabla v_{m\tau}\|^2 \right) d\tau \\ &+ \frac{q-2}{2q} \left(\|\nabla u_m\|^2 + \|\nabla v_m\|^2 \right) + \frac{q-2\gamma-2}{2q\gamma+2q} \left(\|\nabla u_m\|^2 + \|\nabla v_m\|^2 \right)^{\gamma+1} \\ &+ \frac{1}{q^2} \left(\|u_m\|_q^q + \|v_m\|_q^q \right) \\ <\ & d. \end{aligned} \tag{3.9}$$

For a sufficiently large m and $0 \leq t < \infty$, (3.9) gives

$$\begin{aligned} \int_0^t \|u_{m\tau}\|^2 d\tau &< d, \\ \int_0^t \|v_{m\tau}\|^2 d\tau &< d, \end{aligned}$$

$$\begin{aligned} \int_0^t \|\nabla u_{m\tau}\|^2 d\tau &< d, \\ \int_0^t \|\nabla v_{m\tau}\|^2 d\tau &< d, \end{aligned}$$

$$\begin{aligned} \|\nabla u_m\|^2 &< \frac{2q}{q-2} d, \\ \|\nabla v_m\|^2 &< \frac{2q}{q-2} d, \end{aligned}$$

then we have

$$\begin{aligned} &u_{mt} \text{ and } v_{mt} \text{ are bounded in } L^\infty(0,\infty;L^2(\Omega)), \\ &u_m \text{ and } v_m \text{ are bounded in } L^\infty(0,\infty;H_0^1(\Omega)), \\ &u_{mt} \text{ and } v_{mt} \text{ are bounded in } L^2(0,\infty;L^2(\Omega)). \end{aligned}$$

Using the Sobolev embedding inequality

$$\|\nabla u\|^2 + \|\nabla v\|^2 \geq C_*^2 \left(\|u\|_{q+1}^2 + \|v\|_{q+1}^2 \right),$$

for $0 \leq t < \infty$, we have

$$\begin{aligned} \int_\Omega |u_m|^q \ln |u_m|\, dx + \int_\Omega |v_m|^q \ln |v_m|\, dx &\leq \|u_m\|_{q+1}^{q+1} + \|v_m\|_{q+1}^{q+1} \\ &\leq C_*^{q+1} \left(\|u_m\|^{q+1} + \|v_m\|^{q+1} \right) \\ &< \left(\frac{2q}{q-2} d \right)^{\frac{q+1}{2}}. \end{aligned} \tag{3.10}$$

Integrating (3.2) and (3.3) with respect to t, for $0 \leq t < \infty$, we get

$$\begin{aligned} &\int_\Omega u_t(t) w dx + \int_\Omega \nabla u(t) \nabla w dx \\ &+ \int_\Omega \left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \right)^\gamma \nabla u(t) \nabla w dx + \int_\Omega \nabla u_t(t) \nabla w dx \\ = &\int_\Omega |u(t)|^{q-2} u(t) \ln |u(t)|\, w dx, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\int_\Omega v_t(t) w dx + \int_\Omega \nabla v(t) \nabla w dx \\ &+ \int_\Omega \left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \right)^\gamma \nabla v(t) \nabla w dx + \int_\Omega \nabla v_t(t) \nabla w dx \\ = &\int_\Omega |v(t)|^{q-2} v(t) \ln |v(t)|\, w dx, \end{aligned} \tag{3.12}$$

for all $w \in H_0^1(\Omega)$ and for almost every $t \in [0,\infty]$. So, (u,v) is a desired solution of problem (1.1).

Now, we discuss the decay results.

Since $(u_0, v_0) \in \mathcal{W}$, similar to the first case, we obtain $(u,v) \in \mathcal{W}$ for any $t \in [0,\infty)$. By (2.3), (3.11) and (3.12), we get

$$\begin{aligned} J(u_0,v_0) &> J(u,v) \\ &= \frac{1}{q}I(u,v)+\frac{q-2}{2q}\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ &\quad +\frac{q-2\gamma-2}{2q\gamma+2q}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1}+\frac{1}{q^2}\left(\|u\|_q^q+\|v\|_q^q\right), \\ &\geq \frac{q-2}{2q}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\frac{q-2\gamma-2}{2q\gamma+2q}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1} \\ &\quad +\frac{1}{q^2}\left(\|u\|_q^q+\|v\|_q^q\right). \end{aligned} \tag{3.13}$$

For $I(u,v)>0$, (2.4) and Lemma 2.2, there exists a $\lambda^*>1$ such that $I(\lambda^*u,\lambda^*v)=0$. We obtain

$$\begin{aligned} d &\leq J(\lambda^*u,\lambda^*v) \\ &= \frac{1}{q}I(\lambda^*u,\lambda^*v)+\frac{q-2}{2q}\left(\|\nabla(\lambda^*u)\|^2+\|\nabla(\lambda^*v)\|^2\right) \\ &\quad +\frac{q-2\gamma-2}{2q\gamma+2q}\left(\|\nabla(\lambda^*u)\|^2+\|\nabla(\lambda^*v)\|^2\right)^{\gamma+1}+\frac{1}{q^2}\left(\|\lambda^*u\|_q^q+\|\lambda^*v\|_q^q\right) \\ &\leq (\lambda^*)^q\liminf_{k\to\infty}\left[\frac{q-2}{2q}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\frac{q-2\gamma-2}{2q\gamma+2q}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^{\gamma+1}\right] \\ &\quad +(\lambda^*)^q\liminf_{k\to\infty}\frac{1}{q^2}\left(\|u\|_q^q+\|v\|_q^q\right). \end{aligned} \tag{3.14}$$

Using (3.13) and (3.14), we have

$$d\leq(\lambda^*)^qJ(u_0,v_0),$$

which yields that

$$\lambda^*\geq\left(\frac{d}{J(u_0,v_0)}\right)^{\frac{1}{q}}. \tag{3.15}$$

It follows from (2.2) that

$$\begin{aligned}
0 &= I(\lambda^* u, \lambda^* v) \\
&= \left(\|\nabla(\lambda^* u)\|^2 + \|\nabla(\lambda^* v)\|^2\right) + \left(\|\nabla(\lambda^* u)\|^2 + \|\nabla(\lambda^* v)\|^2\right)^{\gamma+1} \\
&\quad - \left(\int_\Omega |\lambda^* u|^q \ln|\lambda^* u|\,dx + \int_\Omega |\lambda^* v|^q \ln|\lambda^* v|\,dx\right) \\
&= (\lambda^*)^2 \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + (\lambda^*)^{2(\gamma+1)} \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&\quad -(\lambda^*)^q \left(\int_\Omega |u|^q \ln|u|\,dx + \int_\Omega |v|^q \ln|v|\,dx\right) - (\lambda^*)^q \ln\lambda^* \left(\int_\Omega |u|^q\,dx + \int_\Omega |v|^q\,dx\right) \\
&= (\lambda^*)^q I(u,v) + (\lambda^*)^2 \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + (\lambda^*)^{2(\gamma+1)} \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&\quad -(\lambda^*)^q \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) - (\lambda^*)^q \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&\quad -(\lambda^*)^q \ln\lambda^* \left(\int_\Omega |u|^q\,dx + \int_\Omega |v|^q\,dx\right) \\
&= (\lambda^*)^q I(u,v) + \left[(\lambda^*)^2 - (\lambda^*)^q\right] \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\
&\quad + \left[(\lambda^*)^{2(\gamma+1)} - (\lambda^*)^q\right] \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&\quad -(\lambda^*)^q \ln\lambda^* \left(\int_\Omega |u|^q\,dx + \int_\Omega |v|^q\,dx\right).
\end{aligned} \tag{3.16}$$

Using (3.15) and (3.16), we have

$$\begin{aligned}
(\lambda^*)^q I(u,v) &= \left[(\lambda^*)^q - (\lambda^*)^2\right] \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \left[(\lambda^*)^q - (\lambda^*)^{2(\gamma+1)}\right] \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\gamma+1} \\
&\quad +(\lambda^*)^q \ln\lambda^* \left(\int_\Omega |u|^q\,dx + \int_\Omega |v|^q\,dx\right) \\
&\geq \left[(\lambda^*)^q - (\lambda^*)^2\right] \left(\|\nabla u\|^2 + \|\nabla v\|^2\right),
\end{aligned}$$

which implies that

$$I(u,v) \geq \left[1 - (\lambda^*)^{2-q}\right] \left(\|\nabla u\|^2 + \|\nabla v\|^2\right). \tag{3.17}$$

It follows from (3.15) and (3.17) that

$$\begin{aligned}
I(u,v) &\geq \left[1 - \left(\frac{d}{J(u_0,v_0)}\right)^{\frac{2-q}{q}}\right] \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\
&\geq C_1 \left[1 - \left(\frac{d}{J(u_0,v_0)}\right)^{\frac{2-q}{q}}\right] \left(\|u\|^2 + \|v\|^2\right),
\end{aligned} \tag{3.18}$$

where C_1 is constant. Also, by (3.18), we obtain

$$\begin{aligned} I(u(t)) &\geq \frac{1}{2}\left[1-\left(\frac{d}{J(u_0,v_0)}\right)^{\frac{2-q}{q}}\right]\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ &\quad +\frac{1}{2}C_1\left[1-\left(\frac{d}{J(u_0,v_0)}\right)^{\frac{2-q}{q}}\right]\left(\|u\|^2+\|v\|^2\right) \\ &\geq C_2\left(\|\nabla u\|^2+\|\nabla v\|^2+\|u\|^2+\|v\|^2\right) \\ &= C_2\left(\|u(t)\|^2_{H_0^1(\Omega)}+\|v(t)\|^2_{H_0^1(\Omega)}\right), \end{aligned} \tag{3.19}$$

where

$$C_2=\max\left\{\frac{1}{2}\left[1-\left(\frac{d}{J(u_0,v_0)}\right)^{\frac{2-q}{q}}\right],\frac{C_1}{2}\left[1-\left(\frac{d}{J(u_0,v_0)}\right)^{\frac{2-q}{q}}\right]\right\}.$$

Integrating the $I(u,v)$ with respect to s over (t,T), we obtain

$$\begin{aligned} \int_t^T I(u,v)ds &= -\int_t^T\int_\Omega\left(u_s(s)u(s)+v_s(s)v(s)\right)dxds \\ &\quad -\int_t^T\int_\Omega\left(\nabla u_s(s)\nabla u(s)+\nabla v_s(s)\nabla v(s)\right)dxds \\ &= \frac{1}{2}\left(\|u(t)\|^2_{H_0^1(\Omega)}+\|v(t)\|^2_{H_0^1(\Omega)}\right)-\frac{1}{2}\left(\|u(T)\|^2_{H_0^1(\Omega)}+\|v(T)\|^2_{H_0^1(\Omega)}\right) \\ &\leq C_3\left(\|u(t)\|^2_{H_0^1(\Omega)}+\|v(t)\|^2_{H_0^1(\Omega)}\right). \end{aligned} \tag{3.20}$$

From (3.19) and (3.20), we have

$$\int_t^T C_2\left(\|u(t)\|^2_{H_0^1(\Omega)}+\|v(t)\|^2_{H_0^1(\Omega)}\right)ds\leq C_3\left(\|u(t)\|^2_{H_0^1(\Omega)}+\|v(t)\|^2_{H_0^1(\Omega)}\right) \text{ for all } t\in[0,T]. \tag{3.21}$$

Let $T\to+\infty$ in (3.21), we can get

$$\int_t^\infty\left(\|u(t)\|^2_{H_0^1(\Omega)}+\|v(t)\|^2_{H_0^1(\Omega)}\right)ds\leq C_4\left(\|u(t)\|^2_{H_0^1(\Omega)}+\|v(t)\|^2_{H_0^1(\Omega)}\right),$$

where $C_4=\frac{C_3}{C_2}$. From Lemma 2.5, we have

$$\|u(t)\|^2_{H_0^1(\Omega)}+\|v(t)\|^2_{H_0^1(\Omega)}\leq\left(\|u(0)\|^2_{H_0^1(\Omega)}+\|v(0)\|^2_{H_0^1(\Omega)}\right)e^{1-\frac{t}{C_4}}.$$

The above inequality implies that the solution (u,v) decays exponentially. Q.E.D.

References

[1] I. Bialynicki-Birula and J. Mycielski, *Nonlinear wave mechanics*, Annals of Physics **100(1-2)** (1976) 62-93.

[2] P. Gorka, *Logarithmic Klein-Gordon equation*, Acta Physica Polonica B **40(1)** (2009) 59-66.

[3] H. Chen and S. Tian, *Initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity*, Journal of Differential Equations **258** (2015) 4424-4442.

[4] H. Chen, P. Luo and G. Liu, *Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity*, Journal of Mathematical Analysis and Applications **422(1)** (2015) 84-98.

[5] J. Peng and J. Zhou, *Global existence and blow-up of solutions to a semilinear heat equation with logarithmic nonlinearit*y, Applicable Analysis **100(13)** (2021) 2804-2824.

[6] L.C. Nhan and L.X. Truong, *Global solution and blow-up for a class of pseudo p-Laplacian evolution equations with logarithmic nonlinearity*, Computers & Mathematics with Applications, **73** (2017) 2076-2091.

[7] Y. He, H. Gao and H. Wang, *Blow-up and decay for a class of pseudo-parabolic p-Laplacian equation with logarithmic nonlinearity*, Computers & Mathematics with Applications **75** (2018) 459-469.

[8] Y. Cao and C. Liu, *Initial boundary value problem for a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity*, Electronic Journal of Differential Equations **116** (2018) 1-19.

[9] E. Pişkin and T. Cömert, *Blow-up of solutions for a parabolic Kirchhoff type equation with logarithmic nonlinearity*, Gulf Journal of Mathematics **9(2)** 2020 21-30.

[10] Y. Yang, J. Li and T. Yu, *Qualitative analysis of solutions for a class of Kirchhoff equation with linear strong damping term, nonlinear weak damping term and power-type logarithmic source term*, Applied Numerical Mathematics **14** (2019) 263-285.

[11] X. Wang, Y.Chen, Y. Yang, J. Li and R. Xu, *Kirchhoff type system with lineer weak damping and logarithmic nonlinearities*, Nonlinear Analysis **188** (2019) 475-499.

[12] T. Cömert and E. Pişkin, *Blow up at infinity of weak solutions for a higher-order parabolic equation with logarithmic nonlinearity*, Journal of Universal Mathematics **4(2)** (2021) 118-127.

[13] T. Cömert and E. Pişkin, *Global Existence and Exponential Decay of Solutions for Higher-Order Parabolic Equation with Logarithmic Nonlinearity*, Miskolc Mathematical Notes (in press).

[14] H. Ding and J. Zhou, *Global existence and blow-up for a parabolic problem of Kirchhoff type with logarithmic nonlinearity*, Applied Mathematics & Optimization **83** (2021) 1651-1707.

[15] Y. Han, C. Cao and P. Sun, *A p-Laplace equation with logarithmic nonlinearity at high initial energy level*, Acta Applicandae Mathematicae **164** (2019) 155-164.

[16] N. Irkıl and E. Pişkin, *Global existence and decay of solutions for a higher-order Kirchhoff-type systems with logarithmic nonlinearities*, Quaestiones Mathematicae, (in press).

[17] P. Li and C. Liu, *A class of fourth-order parabolic equation with logarithmic nonlinearity*, Journal of Inequalities and Applications **328** (2018) 1-21.

[18] E. Pişkin, S. Boulaaras and N. Irkıl, *Qualitative analysis of solutions for the p-Laplacian hyperbolic equation with logarithmic nonlinearity*, Mathematical Methods in the Applied Sciences **44** (2021) 5654-4672.

[19] E. Pişkin and T. Cömert, *Qualitative analysis of solutions for a parabolic type Kirchhoff equation with logarithmic nonlinearity*, Open Journal of Discrete Applied Mathematics **4(2)** (2021) 1-10.

[20] E. Pişkin and Z. Çalışır, *Decay and blow up at infinite time of solutions for a logarithmic Petrovsky equation*, Tbilisi Mathematical Journal **13(4)** (2020) 113-127.

[21] E. Pişkin and N. Irkıl, *Well-posedness results for a sixth-order logarithmic Boussinesq equation*, FLOMAT, **33(13)** (2019) 3985-4000.

[22] E. Pişkin and H. Yüksekkaya, *Local existence of solutions for a p-Laplacian type equation with delay term and logarithmic nonlinearity*, Tbilisi Mathematical Journal **SI(7)** (2021) 77-93.

[23] E. Pişkin and H. Yüksekkaya, *Blow-up of solutions for a logarithmic quasilinear hyperbolic equation with delay term*, Journal of Mathematical Analysis, **12(1)** (2021) 56-64.

[24] H. Yüksekkaya, E. Pişkin, S.M. Boulaaras, B.B. Cherif and S.A. Zubair, *Existence, Nonexistence, and Stability of Solutions for a Delayed Plate Equation with the Logarithmic Source*, Advances in Mathematical Physics (2021) 1-11.

[25] P. Martinez, *A new method to obtain decay rate estimates for dissipative systems*, ESAIM: Control, Optimisation and Calculus Variations **4** (1999) 419-444.